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Cawley's Counterexample to Dirac's Conjecture as a Curved Spacetime

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Cawley's counterexample Lagrangian to Dirac's conjecture on dynamical systems is modified to a line element in curved spacetime, and the energy-momentum tensor corresponding to such a spacetime is found. The spacetime obtained satisfies the Einstein field equations and describes a three-dimensional matter-filled universe. It is further shown that such a universe cannot be filled up with other sources, such as a perfect fluid, a scalar field, or an electromagnetic field, without violating the Einstein field equations.

1. INTRODUCTION

The ambiguity in the Hamiltonian for systems with constraints was discussed by Cawley (1979), who also gave a counterexample to a conjecture by Dirac (1950, 1958, 1964) for the identification of the Hamiltonian of the physical system. Subsequently, it was shown by Frenkel that Cawley's example still leaves open the possibility that Dirac's "test" always provides all the gauge generators, and an example in which both Dirac's conjecture and test fail was given by him (Frenkel, 1980; Cawley, 1980). Because of the great interest in the problem of constraints in classical dynamical systems, Dirac's theory and the counterexamples have drawn attention and stimulated much discussions in the literature (Dominici and Gomis, 1980; Hojman, 1980; Castellani, 1982; Kamimura, 1982; Sugano and Kamo, 1982; Sundermeyer, 1982; diStefano, 1983; Gotay, 1983; Kaptanoglu, 1983; Skinner and Rusk, 1983; Sugano and Kimura, 1983; Costa, Girotti, and Simoes, 1985).

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In this paper we construct a curved spacetime out of Cawley's (1985) Lagrangian and find the corresponding energy-momentum tensor, the source of the curvature. In Section 2 we present our metric and write down the Einstein field equations corresponding to it. In Section 3 the energy-momentum tensor corresponding to this curved spacetime is shown to be that of matter. Other energy-momentum tensors are then discussed in Section 4 and shown to be unsuitable as sources to such a spacetime. The last section is devoted to concluding remarks.

2. THE CURVED SPACETIME METRIC

Changing notation, and adding the appropriate term to the temporal component, Cawley's (1985) Lagrangian can be written in the form of the three-dimensional line element

$$ds^2 = f(x, y)\, dt^2 - 2\, dx\, dy \tag{1}$$

where

$$f(x, y) = 1 + 2kxy - \lambda y^2 \tag{2}$$

Here k is a constant, λ is a parameter taking the values from $-\infty$ to $+\infty$, and x and y are "Cartesian" coordinates. The metric (1) has a singularity when $f(x, y) = 0$, that is, at the two hypersurfaces

$$y_\pm = [kx \pm (k^2x^2 + \lambda)^{1/2}]/\lambda \tag{3}$$

Using the notation $x^0 = t$, $x^1 = x$, $x^2 = y$, one then has for the metric components

$$g_{\mu\nu} = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} f^{-1} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \tag{4}$$

along with

$$(-g)^{1/2} = f^{1/2}, \qquad g = \det g_{\mu\nu} \tag{5}$$

The nonvanishing components of the Christoffel symbol are

$$\begin{aligned} \Gamma^0_{01} &= ky/f, \qquad \Gamma^0_{02} = (kx - \lambda y)/f \\ \Gamma^1_{00} &= kx - \lambda y, \qquad \Gamma^2_{00} = ky \end{aligned} \tag{6}$$

The Ricci tensor is given by $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$, where $R^\alpha_{\mu\beta\nu}$ is the Riemann tensor (Carmeli, 1977)

$$R_{\mu\nu} = (-g)^{-1/2}[(-g)^{1/2}\Gamma^\alpha_{\mu\nu}]_{,\alpha} - [\ln(-g)^{1/2}]_{,\mu\nu} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} \tag{7}$$

where a comma denotes partial differentiation, $f_{,\alpha} = \partial f/x^{\alpha}$, $\alpha = 0, 1, 2$. The components of the Ricci tensor are then given by

$$\begin{aligned} R_{00} &= 2k(1+kxy)/f \\ R_{01} &= R_{02} = 0 \\ R_{11} &= k^2y^2/f^2 \\ R_{12} &= -k(1+kxy)/f^2 \\ R_{22} &= (\lambda + k^2x^2)/f^2 \end{aligned} \tag{8}$$

and the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ by

$$R = 4k(1+kxy)/f^2 \tag{9}$$

The contravariant Ricci tensor is then given by

$$\begin{aligned} R^{00} &= 2k(1+kxy)/f^3 \\ R^{01} &= R^{02} = 0 \\ R^{11} &= (\lambda + k^2x^2)/f^2 \\ R^{12} &= -k(1+kxy)/f^2 \\ R^{22} &= k^2y^2/f^2 \end{aligned} \tag{10}$$

whereas the components of the Einstein tensor $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ are given by

$$\begin{aligned} G^{00} &= G^{01} = G^{02} = 0 \\ G^{11} &= (\lambda + k^2x^2)/f^2 \\ G^{12} &= k(1+kxy)/f^2 \\ G^{22} &= k^2y^2/f^2 \end{aligned} \tag{11}$$

Using the Einstein field equations, we then find that the energy-momentum tensor $T^{\mu\nu}$ is given by

$$f^{-2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda + k^2y^2 & k(1+kxy) \\ 0 & k(1+kxy) & k^2y^2 \end{pmatrix} = \kappa T^{\mu\nu} \tag{12}$$

where κ is the Einstein gravitational constant (in the next sections it will be taken as unity).

In the next section we will find the energy-momentum tensor that provides the sources for this metric and show that it is that one describing matter.

3. THE MATTER ENERGY-MOMENTUM TENSOR

In the last section we constructed a line element out of Cawley's Lagrangian and wrote down the Einstein field equations corresponding to it. As is seen from equation (12), the spacetime obtained is not empty (namely nonvacuum), and the question arises as to what energy-momentum tensor (source) this spacetime corresponds. It is worthwhile mentioning here that the situation is similar to those of the nonvacuum nonstationary Vaidya and Kerr metrics, where one has the metrics at hand and the corresponding energy-momentum tensors are interpreted via the Einstein field equations accordingly (Carmeli and Kaye, 1977; Carmeli, 1982).

In this section we will assume that the energy-momentum tensor is that describing matter and is given by (Carmeli, 1982)

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu} + p^{\mu\nu} \tag{13}$$

where ρ is the mass density, u^{α} is the "four"-velocity $u^{\alpha} = dx^{\alpha}/ds$ of the individual particles, and $p^{\mu\nu}$ is the stress tensor (the speed of light is taken as unity). We now use this expression for the energy-momentum tensor in equation (12) and obtain

$$\begin{aligned}
&\rho u^0 u^0 + p^{00} = 0\\
&\rho u^0 u^1 + p^{01} = 0\\
&\rho u^0 u^2 + p^{02} = 0\\
&\rho u^1 u^1 + p^{11} = (\lambda + k^2 x^2)/f^2\\
&\rho u^1 u^2 + p^{12} = k(1 + kxy)/f^2\\
&\rho u^2 u^2 + p^{22} = k^2 y^2/f^2
\end{aligned} \tag{14}$$

where

$$f = 1 + 2kxy - \lambda y^2 \tag{15}$$

The unknown variables in (14) are, of course, the mass density ρ, the three components u^0, u^1, u^2 of the velocity, and the six components p^{00}, p^{01}, p^{02}, p^{11}, p^{12}, p^{22} of the stress tensor. To these unknown variables we have the six equations (14), the three conservation laws (continuity equations)

$$T^{\mu\nu}_{;\nu} = 0 \tag{16}$$

and the normalization condition $u_{\alpha} u^{\alpha} = 1$. In (16) a semicolon denotes covariant differentiation.

A simple calculation then shows that the conservation laws (16) give

$$\begin{aligned}
&\tilde{\rho}_x u^1 u^1 + \tilde{p}^{11}_x + \tilde{\rho}_y u^1 u^2 + \tilde{p}^{12}_y = 0\\
&\tilde{\rho}_x u^2 u^1 + \tilde{p}^{21}_x + \tilde{\rho}_y u^2 u^2 + \tilde{p}^{22}_y = 0
\end{aligned} \tag{17}$$

where $\tilde{\rho}_x = \partial\tilde{\rho}/\partial x$, $\tilde{\rho}_y = \partial\tilde{\rho}/\partial y$, etc., and

$$\tilde{\rho} = f^{1/2}\rho, \qquad \tilde{p}^{mn} = f^{1/2} p^{mn} \qquad (m, n = 1, 2) \tag{18}$$

The third equation of (16) yields $0 \equiv 0$. Equations (14) and (17) and the condition $u_\alpha u^\alpha = 1$ determine uniquely the ten unknown variables of the energy-momentum tensor up to one undetermined degree of freedom.

In the next section we will try three other energy-momentum tensors as sources for our metric, all of which will be shown to be unsatisfactory.

4. OTHER ENERGY-MOMENTUM TENSORS

In the last section we "matched" an energy-momentum tensor to the metric presented in Section 2. In this section we will try other energy-momentum tensors.

Let us first assume that the matter consists of a perfect fluid, namely, one whose pressure is isotropic. The stress tensor can then be expressed as

$$p^{\mu\nu} = p(u^\mu u^\nu - g^{\mu\nu}) \tag{19}$$

where p is the pressure. We then have the field equations

$$\begin{aligned}
(\rho+p)u^0u^0 - p/f &= 0 \\
(\rho+p)u^0u^1 &= 0 \\
(\rho+p)u^0u^2 &= 0 \\
(\rho+p)u^1u^1 &= (\lambda + k^2x^2)/f^2 \\
(\rho+p)u^1u^2 + p &= k(1+kxy)/f^2 \\
(\rho+p)u^2u^2 &= k^2y^2/f^2
\end{aligned} \tag{20}$$

To satisfy these equations, and assuming that $\rho \neq 0$, one sees that both the pressure and u^0 should vanish,

$$p = 0, \qquad u^0 = 0 \tag{21}$$

From the last three equations of (20) one then obtains the condition

$$(\lambda + k^2x^2)ky^2 = k(1+kxy)^2 \tag{22}$$

which yields

$$f = 0. \tag{23}$$

Thus, such an energy-momentum tensor should be excluded.

We next try an energy-momentum tensor describing a scalar field ϕ,

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - m^2 g_{\mu\nu}\phi \tag{24}$$

A simple calculation then gives the field equations

$$\begin{aligned}\phi_t^2 - m^2\phi f &= 0\\ \phi_t\phi_x &= 0\\ \phi_t\phi_y &= 0\\ \phi_x^2 &= k^2y^2/f^2\\ \phi_x\phi_y + m^2\phi &= k(1+kxy)/f^2\\ \phi_y^2 &= (\lambda + k^2x^2)/f^2\end{aligned} \tag{25}$$

where use has been made of the notation $\phi_t = \partial\phi/\partial t$, $\phi_x = \partial\phi/\partial x$, and $\phi_y = \partial\phi/\partial y$. In analogy to the previous case, one sees that

$$m = 0, \qquad \phi_t = 0 \tag{26}$$

and the field equations (25) again lead to the singularity condition (23). Thus, the energy-momentum tensor (24) should also be excluded.

We will try another possibility, that the energy-momentum tensor describes an electromagnetic field,

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F_{\mu\alpha}F^{\alpha}_{\ \nu}\right) \tag{27}$$

Using the Minkowskian notation

$$F_{\mu\nu}^{\mathrm{Mink}} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & H \\ E_y & -H & 0 \end{pmatrix} \tag{28}$$

where E and H are the electric and magnetic fields, we then obtain for $T_{\mu\nu}$ in our coordinate system

$$T_{\mu\nu} = \frac{1}{4\pi}\begin{pmatrix} E_xE_y - H^2f/2 & -E_xH & E_yH \\ -HE_x & -E_x^2/f & -H^2/2 \\ HE_y & -H^2/2 & H^2 - E_y^2/f \end{pmatrix} \tag{29}$$

Using the field equations, we again obtain contradictions, and thus the electromagnetic field tensor should also be excluded.

Hence the source of the metric (1) cannot be a perfect fluid, a scalar field, or an electromagnetic field.

5. CONCLUDING REMARKS

We conclude from the analyses of Sections 3 and 4 that the appropriate energy-momentum tensor to the metric presented in Section 2 is that describing matter, and that the standard energy-momentum tensors representing a perfect fluid, a Klein-Gordon field, or an electromagnetic field cannot be the source for such a spacetime. As has been pointed out before, our analysis is similar to those used for the nonstationary nonvacuum Vaidya and Kerr metrics. However, while in the Vaidya and the nonstationary Kerr metrics the sources are obviously radiative fields, the source here is just matter.

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